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Simon's conjecture for two-bridge knots

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It is conjectured that for each knot K in S^3, the fundamental group of its complement surjects onto only finitely many distinct knot groups. Applying character variety theory we obtain an affirmative solution of the conjecture for a class of small knots that includes two-bridge knots.

1. Introduction

In this paper all knots and links are in S^3. For a knot (link) K, we often simply call the fundamental group of $S^3 \setminus K$, *the group of* K or *the knot (link) group.* Let K be a non-trivial knot. Simon's Conjecture (see [16, Problem 1.12(D)]) asserts the following:

Conjecture 1.1. $\pi_1(S^3 \setminus K)$ *surjects onto only finitely many distinct knot groups.*

Although this conjecture dates back to the 1970s, and has received considerable attention recently (see [2, 3, 12, 20, 22, 23, 27, 28] to name a few), little by way of general results appears to be known. Conjecture 1.1 is easily seen to hold for torus knots (we give the proof in Section 3.1). In [2], the conjecture is established under the assumption that the epimorphisms are non-degenerate in the sense that the longitude of K is sent to a non-trivial peripheral element under the epimorphism. In particular, this holds in the case when the homomorphism is induced by a mapping of non-zero degree.

Since any knot group is the homomorphic image of the group of a hyperbolic knot (see for example [15]), it is sufficient to prove the conjecture for hyperbolic knots. The main result of this paper is the following, and is the first general result for a large class of hyperbolic knots.

Theorem 1.2. *Conjecture* 1.1 *holds for all two-bridge knots.*

Indeed, using [2, 3, 10, 12], one can say more about the nature of the homomorphisms in Theorem 1.2. In particular, we establish:

Corollary 1.3. *Let K be a two-bridge hyperbolic knot, and K' a non-trivial knot. If there is an epimorphism $\varphi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$, then φ is induced by a map $f : S^3 \setminus K \to S^3 \setminus K'$ of non-zero degree. Furthermore, K' is necessarily a two-bridge knot.*

As we discuss in Section 4.2, a strengthening of Corollary 1.3 holds, where, epimorphism is replaced by *a virtual epimorphism* (see Section 4.2 for more details).

We will also prove some general results towards Conjecture 1.1 for a larger class of knots satisfying certain conditions. These results will be used in proving Theorem 1.2 and Corollary 1.3. Recall that a compact oriented three-manifold N is called *small* if N contains no closed embedded essential surface; and a knot (link) $K \subset S^3$ is called *small* if the exterior $E(K)$ is small. The fact that two-bridge knots are small is proved in [13].

Theorem 1.4. *Let L be a small hyperbolic link of n components. Then $\pi_1(S^3 \setminus L)$ surjects onto only finitely many groups of hyperbolic links of n components.*

Note that a knot is either a torus knot, or a hyperbolic knot, or a satellite knot. In particular, it follows from Theorem 1.4, that if K is a small hyperbolic knot, then $\pi_1(S^3 \setminus K)$ surjects onto only finitely many hyperbolic knot groups. As we discuss below, it is easy to establish using the Alexander polynomial, that any knot group surjects onto only finitely many distinct torus knot groups.

It is perhaps tempting at this point to think that there cannot be an epimorphism from a small knot group to the group of a satellite knot, therefore Conjecture 1.1 holds for small knots. This does not seem so easy to exclude and motivates the following more general question:

Question 1.5. *Does there exist a small knot $K \subset S^3$ such that $\pi_1(S^3 \setminus K)$ surjects onto $\pi_1(S^3 \setminus K')$, and $S^3 \setminus K'$ contains a closed embedded essential surface?*

When the target is the fundamental group of a satellite knot, we will prove that if such a homomorphism exists, then the longitude λ of K must be in the kernel. More precisely,

Proposition 1.6. *Let K be a small hyperbolic knot and let K' be a satellite knot. Assume that $\varphi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$ is an epimorphism. Then $\varphi(\lambda) = 1$.*

In order to control the image of the longitude, we introduce the following definition which can be thought as a kind of smallness for the knot in terms of the character variety of longitudinal surgery on K. For a knot $K \subset S^3$ we denote by $K(0)$ the manifold obtained from S^3 by a longitudinal surgery on K.

Definition 1.7. *Let K be a knot. We will say K has Property L if the* $\mathrm{SL}(2, \mathbf{C})$*-character variety of the manifold $K(0)$ contains only finitely many characters of irreducible representations.*

The motivation for this definition is the following result.

Proposition 1.8. *If a hyperbolic knot K has Property* L*, then for any nontrivial knot K' and epimorphism $\varphi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$,* $\ker \varphi$ *does not contain the longitude of K.*

The following result is thus a consequence of Theorem 1.4 and Propositions 1.6 and 1.8:

Theorem 1.9. *Let K be a small knot and assume that K has Property* L*. Then Conjecture 1.1 holds for K.*

Our main result (Theorem 1.2) now follows immediately from Theorem 1.9 and the next proposition.

Proposition 1.10. *Let K be a hyperbolic two-bridge knot. Then K has Property* L*.*

Although Property L is framed in terms of the character variety, which can be difficult to understand, there are useful criteria which are sufficient for a small knot to have Property L. The first one will be used to show that Property L holds for two-bridge knots. For the definition of a parabolic representation or of a strict boundary slope see Section 2.

Proposition 1.11. *Let K be a small hyperbolic knot.*

- *If no parabolic representation $\rho : \pi_1(S^3 \setminus K) \to \mathrm{SL}(2, \mathbf{C})$ kills the longitude of K, then Property* L *holds for K.*
- *If the longitude is not a strict boundary slope, then Property* L *holds for K.*

Remarks on Property L. (1) Using Proposition 1.8, it is easy to construct knots that do not have Property L.

For example, using the construction of [14] on a normal generator for a knot group that is not a meridian (which exist in some abundance [2], see [6] for explicit examples), one can construct a hyperbolic knot whose group surjects onto another hyperbolic knot group sending the longitude trivially. In [12], examples are given where the domain knot is small; for example there is an epimorphism of the group of the knot 8_{20} onto the group of the trefoil-knot for which the longitude of 8_{20} is mapped trivially.

(2) Control of the image of the longitude has featured in other work related to epimorphisms between knot groups; for example Property Q^* of Simon (see [29] and also [10, 12]). Indeed, from [12], the property given by Proposition 1.8 can be viewed as an extension of Property Q^* of Simon.

(3) Note that if K and K' are knots with Alexander polynomials $\Delta_K(t)$ and $\Delta_{K'}(t)$, respectively, and $\varphi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$ an epimorphism, then it is well-known that $\Delta_{K'}(t)|\Delta_K(t)$. Thus, simple Alexander polynomial considerations show that any knot group surjects onto only finitely many distinct torus knot groups, and so it is only when the target is hyperbolic or satellite that the assumption of Property L is interesting.

The character variety (as in [3, 22]) is the main algebraic tool that organizes the proofs of the results in this paper. In particular, we make use of the result of Kronheimer and Mrowka [18] which ensures that the $\mathrm{SL}(2,\mathbf{C})$-character variety (and hence the $\mathrm{PSL}(2,\mathbf{C})$-character variety) of any non-trivial knot contains a curve of characters of irreducible representations.

A comment on application of the character variety to Simon's conjecture: As we can and will see from [3, 22] and the present paper, the theory of character varieties is particularly useful in the study of epimorphisms between three-manifolds groups when the domain manifolds are small.

However, comparison of the two results below suggests a possible limitation of applying character variety methods to Simon's conjecture as well as the truth of Simon's conjecture itself: On the one hand, the group of each small hyperbolic link of n components surjects onto only finitely many groups of n component hyperbolic links (Corollary 3.2); while on the other hand, there exist hyperbolic links of two components whose groups surject onto the group of every two bridge link (see the discussion related to Conjecture 5.1).

Organization of the paper: The facts about the character variety that will be used later are presented in Section 2. Results stated for small knots, such as Theorem 1.4, Propositions 1.6, 1.8, and Theorem 1.9, will be proved in Section 3. Results stated for two-bridge knots, such as Theorem 1.2, Corollary 1.3, Proposition 1.10, will be proved in Section 4. Section 5 records more questions, consequences and facts for the character variety and Simon's conjecture that have arisen out of our work.

2. Preliminaries

2.1. Some notation

Throughout, if $L \subset S^3$ is a link we shall let $E(L)$ denote the exterior of L; that is the closure of the complement of a small open tubular neighborhood of L. If $K \subset S^3$ is a knot and $r \in \mathbf{Q} \cup \infty$ a slope, then $K(r)$ will denote the manifold obtained by r-Dehn surgery on K (or equivalently, r-Dehn filling on $E(K)$). Our convention is always that a meridian of K has slope $1/0$ and a longitude $0/1$. A slope r is called a *boundary slope*, if $E(K)$ contains an embedded essential surface whose boundary consists of a non-empty collection of parallel copies of simple closed curves on $\partial E(K)$ of slope r. The longitude of a knot K always bounds a Seifert surface of K, and so is a boundary slope. It is called a *strict boundary slope* if it is the boundary slope of a surface that is not a fiber in a fibration over the circle.

2.2. Standard facts about the character variety

Let G be a finitely generated group. We denote by $X(G)$ (resp. $Y(G)$) the $\mathrm{SL}(2,\mathbf{C})$-character variety (resp. $\mathrm{PSL}(2,\mathbf{C})$-character variety) of G (see [4, 8] for details). If V is an algebraic set, we define the *dimension* of V to be the maximal dimension of an irreducible component of V. We will denote this by $\dim(V)$.

Suppose that G and H are finitely generated groups and $\varphi : G \to H$ is an epimorphism. Then φ defines a map at the level of character varieties $\varphi^* : X(H) \to X(G)$ by $\varphi^*(\chi_\rho) = \chi_{\rho\circ\varphi}$. This map is algebraic, and furthermore is a closed map in the Zariski topology (see [3, Lemma 2.1]). In future we will abbreviate composition of homomorphisms $\varphi \circ \psi$ by $\varphi\psi$.

We make repeated use of the following standard fact.

Lemma 2.1. *Let G and H be as above, then φ^* injects $X(H) \hookrightarrow X(G)$.*

Proof. Suppose $\chi_\rho, \chi_{\rho'} \in X(H)$ with $\varphi^*(\chi_\rho) = \varphi^*(\chi_{\rho'})$. Thus, $\chi_{\rho\varphi}(g) = \chi_{\rho'\varphi}(g)$ for all $g \in G$, and since φ is onto, we deduce that $\chi_\rho(h) = \chi_{\rho'}(h)$ for all $h \in H$. Hence $\chi_\rho = \chi_{\rho'}$. □

We now assume that $D \subset X(G)$ is a component containing the character χ_ρ of an irreducible representation and $D = \varphi^*(C)$ (as noted φ^* is a closed map) for some component $C \subset X(H)$. Then, $\chi_\rho = \varphi^*(\chi_{\rho'})$ for some irreducible representation $\rho' : H \to \mathrm{SL}(2, \mathbf{C})$. By definition, $\varphi^*(\chi_{\rho'}) = \chi_{\rho'\varphi}$, and so since the representations ρ and $\rho'\varphi$ are irreducible, we deduce that the groups $\rho(G)$ and $\rho'\varphi(G) = \rho'(H)$ are conjugate in $\mathrm{SL}(2, \mathbf{C})$. In particular, after conjugating if necessary, the homomorphisms $\rho'\varphi$ and ρ have the same image.

2.3. Existence of irreducible representations of knot groups

When $G = \pi_1(M)$, and M is a compact three-manifold we denote $X(G)$ (resp. $Y(G)$) by $X(M)$ (resp. $Y(M)$). When M is a knot exterior in S^3 we write $X(M) = X(K)$ (resp. $Y(M) = Y(K)$).

Now $X(K)$ (resp. $Y(K)$) always contains a curve of characters corresponding to abelian representations. When K is a hyperbolic knot (i.e., $S^3 \setminus K$ admits a complete hyperbolic structure of finite volume), it is a well-known consequence of Thurston's Dehn surgery theorem (see [9, Proposition 1.1.1]) that there is a so-called *canonical component* in $X(K)$ (resp. $Y(K)$), which is a curve, and contains the character of a faithful discrete representation of $\pi_1(S^3 \setminus K)$. More recently, the work of Kronheimer and Mrowka [18] establishes the following general result (we include a proof of the mild extension of their work that is needed for us).

Theorem 2.2. *Let K be a non-trivial knot. Then $X(K)$ (resp. $Y(K)$) contains a curve for which all but finitely many of its elements are characters of irreducible representations.*

Proof. It suffices to prove Theorem 2.2 for $X(K)$. As the set of reducible characters is Zariski closed in $X(K)$ [8, proof of Corollary 1.4.5], by a result of Thurston (see [8, Proposition 3.2.1]) to find a curve as in the conclusion of Theorem 2.2, it is enough to find an irreducible representation $\rho : \pi_1(E(K)) \to \mathrm{SL}(2, \mathbf{C})$ such that $\rho(\pi_1(\partial E(K))) \not\subset \{\pm I\}$. Note that the latter condition holds for any irreducible representation of $\pi_1(E(K))$.

To find an irreducible representation ρ we proceed as follows. Note that by [18], for any $r \in \mathbf{Q}$ with $|r| \leq 2$, $\pi_1(K(r))$, admits a non-cyclic SU(2)-representation. Take $r = 1$ and suppose that the representation ϕ of

$\pi_1(K(1))$ guaranteed by [18] is reducible as a representation into $\mathrm{SL}(2, \mathbf{C})$. Since $\pi_1(K(1))$ is perfect, it coincides with its commutator subgroup and therefore the trace of any element of the image of ϕ is 2. As I is the only element of $\mathrm{SU}(2)$ with this trace, the image of ϕ is $\{I\}$ which is a contradiction.

Composing ϕ with the epimorphism induced by 1-Dehn surgery on K determines a representation $\rho : \pi_1(E(K)) \to \mathrm{SL}(2, \mathbf{C})$ whose image coincides with that of ϕ. This is the required irreducible representation. $\square$

2.4. $X(K)$ for small hyperbolic knots and p-rep. characters

We now prove some results about the character variety of a small hyperbolic knot. It will be convenient to recall some terminology from [9].

Let $K \subset S^3$ be a knot and $\alpha \in \pi_1(\partial E(K))$. If $X \subset X(K)$ is a component, define the polynomial function

$$f_\alpha : X \to \mathbf{C} \quad \text{by} \quad f_\alpha(\chi_\rho) = \mathrm{tr}^2(\rho(\alpha)) - 4.$$

We first record the following well-known result.

Theorem 2.3. (1) *Let N be a hyperbolic three-manifold with ∂N a union of n tori. If X is an irreducible component of $X(N)$ that contains the character of an irreducible representation, then* $\dim(X)$ *is at least n; moreover* $\dim(X) = n$ *when N is small.*

(2) *Let K be a small hyperbolic knot and μ be a meridian of K. If x is an ideal point of X, then f_μ has a pole at x. In particular, f_μ is non-constant.*

Proof. (1) The dimension of X is at least n by [8, Proposition 3.2.1] and at most n when N is small by [7, Theorem 4.1].

(2) Let x be an ideal point of X and consider

$$I_\mu : X \to \mathbf{C}, \quad I_\mu(\chi_\rho) = \mathrm{tr}(\rho(\mu)).$$

Clearly $f_\mu = I_\mu^2 - 4$, so to prove the lemma it suffices to show that I_μ has a pole at x. Now [9, Proposition 1.3.9] implies that either $I_\mu(x) = \infty$, or μ is a boundary slope, or $I_\alpha(x) \in \mathbf{C}$ for all $\alpha \in \pi_1(\partial E(K))$. The second possibility is ruled out by [9, Theorem 2.0.3], whereas the third is ruled out by the fact that it implies $E(K)$ contains a closed essential surface (cf. the second paragraph of [9, Section 1.6.2]), which contradicts that $E(K)$ is small. $\square$

Note that zeroes of f_α correspond to representations ρ for which α either maps trivially (in $\mathrm{PSL}(2, \mathbf{C})$) or to a parabolic element. In this latter case, it is easy to see that $f_\beta(\chi_\rho) = 0$ for all $\beta \in \pi_1(\partial E(K))$. Following Riley [24], we call such a representation a *parabolic representation* or *p-rep*. We define a character χ_ρ to be a *p-rep character* if ρ is an irreducible representation for which at least one peripheral element is mapped to a parabolic element.

The following proposition will be useful.

Proposition 2.4. *Let K be a small hyperbolic knot and $X \subset X(K)$ an irreducible component that contains the character of an irreducible representation. Then X contains a p-rep character. Indeed, the set of p-rep characters on X is the zero set of f_μ on X.*

Proof. By Theorem 2.3(1), X is a curve. Let $\tilde{X}$ be its smooth projective model. Then $\tilde{X} = X^\nu \cup \mathcal{I}$ where $\nu : X^\nu \to X$ is an affine desingularization and $\mathcal{I}$ is the finite set of ideal points of X. The function f_μ corresponds to a holomorphic map $\tilde{f}_\mu : \tilde{X} \to \mathbf{CP}^1$ (see [8]) where $\tilde{f}_\mu | X^\nu = f_\mu \circ \nu$. Thus Theorem 2.3 implies that $\tilde{f}_\mu$ is non-constant, so it has at least one zero x_0, and also that $x_0 \in X^\nu$. Set $\nu(x_0) = \chi_\rho$. Since X contains an irreducible character, [9, Proposition 1.5.5] implies that we can suppose the image of ρ is non-cyclic. Hence $\rho(\mu) \neq \pm I$ and therefore $\rho(\mu)$ is parabolic. It follows that if $\alpha \in \pi_1(\partial E(K))$, then either $\rho(\alpha)$ is parabolic, or $\rho(\alpha)$ is $\pm I$. Thus, the proof of the proposition will be complete once we establish that ρ is irreducible.

Suppose this were not the case and let R be the four-dimensional component of the representation variety $R(K) = \mathrm{Hom}(\pi_1(E(K)), \mathrm{SL}(2, \mathbf{C}))$ whose image in $X(K)$ equals X (cf. [4, Lemma 4.1; 10, Corollary 1.5.3]). By [9, Proposition 1.5.6] we can suppose $\rho \in R$. Since R is $\mathrm{SL}(2, \mathbf{C})$-invariant [8, Proposition 1.1.1], we can suppose that the image of ρ consists of upper-triangular matrices. Hence consideration of the sequence $\rho_n = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & n \end{pmatrix} \rho \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & n \end{pmatrix}^{-1}$ shows that R contains a representation ρ_0 whose image is diagonal and which sends μ to $\pm I$. Thus $\rho_0(\gamma) = \pm I$ for all $\gamma \in \pi_1(E(K))$. The Zariski tangent space of R at ρ_0 is naturally a subspace of the vector space of one-cocycles $Z^1(\pi_1(E(K)); \mathrm{sl}(2, \mathbf{C})_{Ad \circ \rho_0})$ (see [31]). Since the image of ρ_0 is central in $\mathrm{SL}(2, \mathbf{C})$, $\mathrm{sl}(2, \mathbf{C})_{Ad \circ \rho_0}$ is a trivial $\pi_1(E(K))$-module. It follows that $Z^1(\pi_1(E(K)); \mathrm{sl}(2, \mathbf{C})_{Ad \circ \rho_0}) \cong H^1(\pi_1(E(K)); \mathrm{sl}(2, \mathbf{C})_{Ad \circ \rho_0}) \cong H^1(\pi_1(E(K)); \mathbf{C}^3) \cong \mathbf{C}^3$. Hence the dimension of the Zariski tangent space of R at ρ_0 is at most 3. But this contradicts the fact that R is four-dimensional. Thus ρ must be irreducible.

To complete the proof, simply note that we have shown that each zero of f_μ on a curve component of $X(K)$ containing the character of an irreducible representation is the character of a p-rep. The converse is obvious. □

3. Results for small knots

3.1. Simon's Conjecture for torus knots

In this section we give a quick sketch of the proof that torus knots satisfy Conjecture 1.1 (see also Section 2 of [28]). Here Property L is not needed.

Thus suppose that K is a torus knot, and assume that there exist infinitely many distinct knots K_i and epimorphisms

$$\varphi_i : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K_i).$$

Note that if z generates the center of $\pi_1(S^3 \setminus K)$, then $\varphi_i(z) \neq 1$; otherwise, φ_i factorizes through a homomorphism of the base orbifold group $C_{r,s}$ which is the free product of two cyclic groups of orders r and s for some co-prime integers r and s. This is impossible, since $\pi_1(S^3 \setminus K_i)$ is torsion-free. Thus $\pi_1(S^3 \setminus K_i)$ has non-trivial center, and so is a torus knot group by Burde–Zieschang's characterization of torus knots [5].

However, as mentioned in Section 1, $\Delta_{K_i}(t)$ will be a factor of $\Delta_K(t)$, and so it easily follows that only finitely many of these K_i can be distinct torus knots. This completes the proof. □

Using this result for torus knots, to prove Conjecture 1.1 for small knots, it therefore suffices to deal with the cases where the domain is a hyperbolic knot. That is the case we will consider in the remainder of this section.

3.2. Proof of Theorem 1.4

In this section we will first prove Theorem 1.4. As remarked upon in Section 1, the finiteness of torus knot groups follows from Alexander polynomial considerations. The finiteness of hyperbolic knot group targets follows easily from our next result. Recall that an elementary fact in algebraic geometry is that the number of irreducible components of an algebraic set V is finite, and hence there are only finitely many of any given dimension n.

Theorem 3.1. *Let G be a finitely generated group. Assume that $\dim(X(G)) = n$ and let m denote the number of irreducible components of $X(G)$ of dimension n. Suppose that for $i = 1, \dots, k$, N_i is a hyperbolic three-manifold with incompressible boundary consisting of precisely n torus boundary components, and that G surjects onto $\pi_1(N_i)$. We assume that the N_i's are all non-homeomorphic. Then $k \le m$.*

Proof. Let $\varphi_i : G \to \pi_1(N_i)$ be the surjections for $i = 1, \dots, k$. As discussed in Section 2.1, this induces a closed algebraic map $\varphi_i^* : X(N_i) \hookrightarrow X(G)$ that is injective by Lemma 2.1. Furthermore, if X_i denotes the canonical component of $X(N_i)$, then $\dim(X_i) = n$ by Thurston's Dehn Surgery Theorem.

Suppose that $k > m$. Then there exists $i, j \in \{1, \dots, k\}$, and an irreducible component $X' \subset X(G)$ of dimension at least n such that

$$\varphi_i^*(X_i), \varphi_j^*(X_j) \subset X'.$$

By the injectivity of φ_i^* and the assumption that $\dim(X(G)) = n$ it follows that $\varphi_i^*(X_i)$, $\varphi_j^*(X_j)$ and X' all have dimension n and so as φ_i^* and φ_j^* are closed maps

$$\varphi_i^*(X_i) = X' = \varphi_j^*(X_j).$$

Relabelling for convenience, we set $i, j = 1, 2$. The equality of these varieties implies that for each

$$\chi_{\rho_1} \in X_1, \quad \text{there exists} \quad \chi_{\rho_2'} \in X_2 \quad \text{with} \quad \varphi_1^*(\chi_{\rho_1}) = \varphi_2^*(\chi_{\rho_2'}),$$

and for each

$$\chi_{\rho_2} \in X_2, \quad \text{there exists} \quad \chi_{\rho_1'} \in X_1 \quad \text{with} \quad \varphi_2^*(\chi_{\rho_2}) = \varphi_1^*(\chi_{\rho_1'}).$$

In particular, we can take ρ_1 to be the faithful discrete representation of $\pi_1(N_1)$, and ρ_2 to be the faithful discrete representation of $\pi_1(N_2)$. Since both ρ_1 and ρ_2 are faithful, we have

$$\rho_1(\pi_1(N_1)) \cong \pi_1(N_1) \quad \text{and} \quad \rho_2(\pi_1(N_2)) \cong \pi_1(N_2).$$

Hence from above, this yields representations $\rho_1' : \pi_1(N_1) \to \mathrm{SL}(2, \mathbf{C})$ and $\rho_2' : \pi_1(N_2) \to \mathrm{SL}(2, \mathbf{C})$ which satisfy

$$\rho_2'(\pi_1(N_2)) \cong \pi_1(N_1) \quad \text{and} \quad \rho_1'(\pi_1(N_1)) \cong \pi_1(N_2).$$

Hence, we get *epimorphisms*:

$$\rho_2'\rho_1' : \pi_1(N_1) \to \pi_1(N_1) \quad \text{and} \quad \rho_1'\rho_2' : \pi_1(N_2) \to \pi_1(N_2).$$

It is well-known that the fundamental groups of compact hyperbolic three-manifolds are Hopfian, and so $\rho_2'\rho_1'$ and $\rho_1'\rho_2'$ are isomorphisms. It now follows that ρ_1' must be also an injection, hence $\pi_1(N_1) \cong \pi_1(N_2)$. Since both N_1 and N_2 are complete hyperbolic three-manifolds with finite volume, N_1 and N_2 are homeomorphic by Mostow Rigidity Theorem, which contradicts the assumption that they are non-homeomorphic. □

The most interesting and immediate application of Theorem 3.1 is the following (Theorem 1.4 of Section 1):

Corollary 3.2. *Let L be a small hyperbolic link of n components. Then $\pi_1(S^3 \setminus L)$ surjects onto only finitely many groups of hyperbolic links of n components.*

Proof. The exterior of each link of n-components has a union of n tori as boundary, and for a small hyperbolic link L of n components $\dim(X(L)) = n$ by Theorem 2.3(1). Then the proof follows readily from Theorem 3.1. □

Theorem 3.1 also provides information about the nature of $X(K)$ for possible counterexamples to Conjecture 1.1.

Corollary 3.3. *Suppose $K \subset S^3$ is a hyperbolic knot and assume that $K_i \subset S^3$ is an infinite family of distinct hyperbolic knots for which there are epimorphisms $\varphi_i : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K_i)$. Then $X(K)$ contains an irreducible component of dimension at least 2.*

Proof. If all components have dimension 1, then Theorem 3.1 bounds the number of knots K_i. □

Remark. *Theorem 3.1 can also be formulated for the* $\mathrm{PSL}(2, \mathbf{C})$*-character variety.*

3.3. Satellite targets

In this section we prove Proposition 1.6. Before giving the proof we fix some notation that will be employed in Sections 3.3 and 3.4.

Notation. Let K be a knot, λ be a longitude for K, μ a meridian for K commuting with λ and we denote by P the peripheral subgroup of $\pi_1(S^3 \setminus K)$ generated by them.

Proof of Proposition 1.6. Suppose that K is a small hyperbolic knot, K' is a satellite knot, and that there exists an epimorphism

$$\varphi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K').$$

Suppose that $\varphi(\lambda) \neq 1$. Since a knot group is torsion-free, $\varphi(P)$ is either infinite cyclic or isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$, Assume that the former case holds. Then there is some primitive slope $r = \mu^m \lambda^n$ such that $\varphi(r) = 1$, so φ factors through the fundamental group of $K(r)$. This is impossible, since by assumption, $r \neq \lambda^{\pm 1}$, so $\pi_1(K(r))$ has finite abelianization, and thus cannot surject onto $\pi_1(S^3 \setminus K')$.

Thus we can assume that $\varphi(P) \cong \mathbf{Z} \oplus \mathbf{Z}$. Suppose $f : E(K) \to E(K')$ is a map realizing φ and let $T = \partial E(K)$. Let T' be a JSJ torus of $E(K')$.

By the enclosing property of the JSJ decomposition we may assume that f has been homotoped so that

(1) $f(T) \subset \Sigma$, where Σ is a piece of the JSJ decomposition.

Moreover we can assume that

(2) $f^{-1}(T')$ is a two-sided incompressible surface in $E(K)$; and

(3) $f^{-1}(T')$ has minimum number of components.

Note that $f^{-1}(T')$ cannot be empty, otherwise since T' is a separating torus in $E(K')$, $f(E(K))$ will miss some vertex manifold of $E(K')$, therefore $f_* = \varphi$ cannot be surjective. No component T^* of $f^{-1}(T')$ is parallel to T, otherwise we can push the image of the product bounded by T and T^* across T' to reduce the number of components of $f^{-1}(T')$. Therefore $f^{-1}(T')$ is closed embedded essential surface in $E(K)$. This is false since K is small. $\square$

3.4. Property L

We start by proving Proposition 1.8 which shows that Property L allows control of the image of a longitude under a knot group epimorphism. Notation for a longitude and meridian is that of Section 3.3. We remark that it is here that crucial use is made of [18].

Proof of Proposition 1.8. Let K be a hyperbolic knot and $\varphi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$ an epimorphism. If $\varphi(\lambda) = 1$, then the epimorphism φ factorizes through an epimorphism $\varphi' : \pi_1(K(0)) \to \pi_1(S^3 \setminus K')$. Now Theorem 2.2 provides a curve of characters $C \subset X(K')$ whose generic point is the character of an irreducible representation. By Lemma 2.1, the curve $D = \varphi'^*(C) \subset X(K(0))$ contradicts the Property L assumption. $\square$

Together with Proposition 1.6, we obtain:

Corollary 3.4. *There cannot be an epimorphisim from the group of a small knot having Property* L *onto the group of a satellite knot.*

Now we prove Proposition 1.11 whose content is given by the following Lemmas:

Lemma 3.5. *Let K be a small hyperbolic knot with the property that for any parabolic representation ρ, we have that $\rho(\lambda) \neq 1$. Then Property* L *holds for K.*

Proof. Suppose that $X(K(0))$ contained a curve C of characters of irreducible representations. Then the epimorphism, $\psi : \pi_1(S^3 \setminus K) \to \pi_1(K(0))$ induced by 0-Dehn surgery together with Lemma 2.1, provides a curve $\psi^*(C) = D \subset X(K)$. Proposition 2.4 shows that D contains a p-rep. character χ_θ, and by assumption $\theta(\lambda) \neq 1$. On the other hand, $\chi_\theta = \psi^*(\chi_{\theta'}) = \chi_{\theta'\psi}$ for some $\chi_{\theta'}$ in C. Hence $\theta = \theta'\psi$ up to conjugacy. Since θ factorizes through $\psi : \pi_1(S^3 \setminus K) \to \pi_1(K(0))$, we must have $\theta(\lambda) = 1$ and therefore reach a contradiction. $\square$

The second part of Proposition 1.11 follows from:

Lemma 3.6. *Let K be a small hyperbolic knot. If the longitude is not a strict boundary slope, then Property* L *holds for K.*

Proof. Let K be a small hyperbolic knot whose preferred longitude λ for K is not a strict boundary slope. Assume that the character variety $X(K(0))$ contains a curve of characters C whose generic element is the character of an irreducible representation. The epimorphism $\varphi : \pi_1(S^3 \setminus K) \to \pi_1(K(0))$ and Lemma 2.1 provide a curve component $D = \varphi^*(C) \subset X(K)$. Since $\varphi(\lambda) = 1$, $f_\lambda : D \to \mathbf{C}$ is identically 0. Thus we deduce that λ is a boundary slope detected by any ideal point of D (cf. proof of Theorem 2.3).

Fix an irreducible character $\chi_\rho \in D$. By hypothesis, λ is not a strict boundary slope, so [4, Proposition 4.7(2)] implies that the restriction of ρ to the index 2 subgroup $\tilde{\pi}$ of $\pi_1(S^3 \setminus K)$ has Abelian image. The irreducibility of ρ implies that this image is non-central in $\mathrm{SL}(2, \mathbf{C})$, and as it is normal in the image of ρ, the latter is conjugate into the subgroup of $\mathrm{SL}(2, \mathbf{C})$ of matrices which are either diagonal or have zeroes on the diagonal. Further, the image of $\tilde{\pi}$ conjugates into the diagonal matrices and that of a meridian of K conjugates to a matrix with zeroes on the diagonal. Any such representation of $\pi_1(S^3 \setminus K)$ has image a finite binary dihedral group. As there are only finitely many such characters of $\pi_1(S^3 \setminus K)$ ([17, Theorem 10]), the generic character in D, hence C, is reducible, a contradiction. □

We can now give the proof of our main technical result Theorem 1.9.

Proof of Theorem 1.9. We are supposing that K is a small hyperbolic knot with Property L. By Corollary 3.4, the targets cannot be fundamental groups of satellite knot complements, hence they must be fundamental groups of hyperbolic or torus knot complements. The case of torus knots was dealt with in the "Remarks on Property L" of Section 1. The proof is completed by Theorem 1.4. □

4. Results for two-bridge knots

Given the discussion for torus knots in Section 3.1, it suffices to deal with the case of a hyperbolic two-bridge knot.

4.1. Proof of Theorem 1.2 and Proposition 1.10

As mentioned in the introduction Theorem 1.2 follows from Theorem 1.9 and Proposition 1.10. This proposition is a straightforward consequence of Lemma 3.5 and of the following lemma of Riley (see Lemma 1 of [25]). We have decided to include a proof of this lemma since it is a crucial point.

Lemma 4.1. *Let K be a two-bridge knot. If $\theta : \pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbf{C})$ is a p-rep, then $\theta(\lambda) \neq 1$.*

Proof. We begin by recalling some of the basic set up of p-reps. of two-bridge knot groups (see [24]). Let K be two-bridge of normal form (p, q), so p and q are odd integers such that $0 < q < p$. The case of $q = 1$ is that of two-bridge torus knots. The group $\pi_1(S^3 \setminus K)$ has a presentation

$$< x_1, x_2 \mid w x_1 w^{-1} = x_2^{-1} >, \quad \text{where} \quad x_1, x_2 \quad \text{are meridians and}$$

$w = w(x_1, x_2)$ is given by $x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_1^{\epsilon_{p-2}} x_2^{\epsilon_{p-1}}$. Furthermore, each exponent $\epsilon_j = (-1)^{[jq/p]}$ where $[x]$ denotes the integer part of x, and $\epsilon_j = \epsilon_{p-j}$. Hence $\sigma = \Sigma \epsilon_j$ is even.

The standard form for a p-rep sends the meridians x_1 and x_2 to parabolic elements:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix},$$

for some non-zero algebraic integer y (indeed y is a unit). The relation in the presentation provides a p-rep polynomial $\Lambda(y)$, and all p-reps determine and are determined by solutions to $\Lambda(y) = 0$. The image of w under p-rep has the form

$$W = \begin{pmatrix} 0 & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

with the entries being functions of the variable y. In addition, as is shown in [24], the image of a longitude that commutes with x_1 has the form

$$\begin{pmatrix} 1 & -2g \\ 0 & 1 \end{pmatrix} \quad \text{for some algebraic integer} \quad g = g(y).$$

Indeed, as shown in [24], $g = w_{12} w_{22} + \sigma$. Thus, to prove the lemma we need to show that $g = g(y) \neq 0$.

This is done as follows. First, observe that (mod 2), the matrix W for the two-bridge knot of normal form (p, q) is the same as the matrix W' one obtains from the two-bridge torus knot with normal form $(p, 1)$. Furthermore, the word w in the case of $(p, 1)$ is given as $(x_1 x_2)^n$ with $n = (p-1)/2$ the degree of $\Lambda(y)$. Using this allows for an easy recursive definition of the matrix W' in this case (see Section 5 of [24]); namely define two sequences of polynomials $f_j = f_j(y)$ and $g_j = g_j(y)$, with $f_0(y) = g_0(y) = 1$ and

$$f_{j+1}(y) = f_j(y) + y g_j(y) \quad \text{and} \quad g_{j+1}(y) = f_{j+1}(y) + g_j(y).$$

Then the matrix W' is given by

$$W' = \begin{pmatrix} f_n & g_{n-1} \\ y g_{n-1} & f_{n-1} \end{pmatrix}.$$

In particular, the p-rep condition implies $f_n(y) = 0$. Using the recursive formula, we have $f_n(y) = f_{n-1}(y) + y g_{n-1}(y)$, and the p-rep condition (i.e.,

$f_n(y) = 0$) means that the matrix W' is given by

$$W' = \begin{pmatrix} 0 & g_{n-1} \\ yg_{n-1} & -yg_{n-1} \end{pmatrix},$$

which (mod 2) is

$$W' = \begin{pmatrix} 0 & g_{n-1} \\ yg_{n-1} & yg_{n-1} \end{pmatrix}.$$

We deduce from these comments that $w_{12}w_{22} = -w_{12}w_{21}$ (mod 2). The latter is 1 since it is the determinant of W. As noted above, σ is even, hence, it follows that $g = w_{12}w_{22} + \sigma$ is congruent to 1 (mod 2), and so in particular is not zero as required. □

4.2. Proof of Corollary 1.3

Proof of Corollary 1.3. As before we let μ and λ denote a meridian and a longitude of K. Firstly, we note that if K is a hyperbolic two-bridge knot and $\varphi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$ is an epimorphism, then Proposition 1.6 and Lemma 4.1 combine to show that K' is either a hyperbolic or torus knot.

In the case of K' a hyperbolic knot, since $\varphi(\lambda) \neq 1$, the epimorphism is non-degenerate in the sense of [2], and in particular $\varphi(\mu)$ is a peripheral element of $\pi_1(S^3 \setminus K')$. Hence, [3, Theorem 3.15] applies to show that K' is also a two-bridge knot. Furthermore, as noted in the proof of Corollary 6.5 of [2], the homomorphism φ is induced by a map of non-zero degree.

In the case when K' is a torus knot, $\varphi(\lambda)$ (and therefore also $\varphi(\mu)$), need not be a peripheral element. Suppose that K' is an (r, s)-torus knot and fix a meridian μ' of K'. There is a homomorphism $\psi : \pi_1(S^3 \setminus K') \to C_{r,s}$ (where as in Section 3.1, $C_{r,s}$ denotes the free product of two cyclic groups of orders r and s) and generators a of $\mathbf{Z}/r$ and b of $\mathbf{Z}/s$ for which $\psi(\mu') = ab$. Theorem 2.1 of [10] and the remark following it, shows that one of r or s equals 2, say $r = 2$. In particular K' is a two-bridge torus knot. We finish off this case as we did the previous one using [2] once we show that $\varphi(< \mu, \lambda >)$ is a subgroup of finite index in the peripheral subgroup P' of $\pi_1(S^3 \setminus K')$. To do this, it suffices to show $\varphi(\mu)$ is a meridian of K' since the centralizer of μ' in $\pi_1(S^3 \setminus K')$ is P'.

To that end, Theorem 1.2 of [10] shows that there is an isomorphism $\theta : C_{2,s} \to C_{2,s}$ such that $\theta\psi\varphi(\mu) = ab^m$ for some integer m. Up to inner isomorphism, we can suppose that $\theta(a) = a$ and $\theta(b) = b^k$ for some k coprime with s (see, for example,[11, Theorem 13(1), Corollary 14]). Thus we can assume that $\psi\varphi(\mu) = ab^m$. Now $\varphi(\mu)$ equals $(\mu')^{\pm 1}$ up to multiplication by a commutator, so abelianizing in $C_{2,s}$ shows that $m \equiv \pm 1 \pmod{s}$. Hence $\psi\varphi(\mu) = ab^{\pm 1}$, so up to conjugation in $C_{2,s}$, $\psi\varphi(\mu) = (ab)^{\pm 1} = \psi(\mu')^{\pm 1}$. It follows that $\varphi(\mu) = h^k(\mu')^{\pm 1}$ where $h \in \pi_1(S^3 \setminus K')$ is the fiber class. Since K' is non-trivial, $|s| \geq 3$, and so as h represents $2s$ in $H_1(S^3 \setminus K') \cong \mathbf{Z}$, it must be that $k = 0$. Thus $\varphi(\mu) = (\mu')^{\pm 1}$, which completes the proof. $\square$

We conclude this section with some remarks on the proof of a stronger version of Corollary 1.3. Before stating this result, we recall that if G and H are groups and $\varphi : G \to H$ is a homomorphism, then φ is called a *virtual epimorphism* if $\varphi(G)$ has finite index in H.

Theorem 4.2. *Let K be a two-bridge hyperbolic knot, K' be a non-trivial knot. If there is a virtual epimorphism $\varphi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$, then φ is induced by a map $f : S^3 \setminus K \to S^3 \setminus K'$ of non-zero degree. Furthermore, K' is necessarily a two-bridge knot, and φ is surjective if K' is hyperbolic.*

Sketch of the Proof. Since a subgroup of finite index in a satellite knot group continues to contain an essential $\mathbf{Z} \oplus \mathbf{Z}$ the proof of Proposition 1.6 can be applied to rule out the case of satellite knot groups as targets.

In the case where the targets are hyperbolic, we can deduce that this virtual epimorphism is an epimorphism and we argue as before; briefly, since the peripheral subgroup is mapped to a $\mathbf{Z} \oplus \mathbf{Z}$ in the image, and since K is two-bridge, it follows from [1, Corollary 5] that these image groups are two-bridge knot groups (being generated by two conjugate peripheral elements). However, it is well-known that a two-bridge hyperbolic knot complement has no free symmetries, and so cannot properly cover any other hyperbolic three-manifold (see [26] for example).

When the targets are torus knot groups, standard considerations show that the image of φ is the fundamental group of a Seifert Fiber Space with base orbifold a disc with cone points. Moreover, this two-orbifold group is generated by the images of the two conjugate meridians of K. It is easily seen that this forces the base orbifold to be a disc with two cone points. It now follows from [11, Proposition 17] that the base orbifold group is $C_{2,s}$ where s is odd, and the proof is completed as before. $\square$

Remark. Note that the paper [20] gives a systematic construction of epimorphisms between two-bridge knot groups. In particular, the epimorphisms constructed by the methods of [20] are induced by maps of non-zero degree. Corollary 1.3 and Theorem 4.2 show that in fact *any* (virtual) epimorphism from a two-bridge knot group to any knot group is induced by a map of non-zero degree.

4.3. Minimal manifolds and Simon's Conjecture

The methods of this paper also prove the following strong form of Conjecture 1.1 in certain cases.

Theorem 4.3. *Suppose $K \subset S^3$ is a hyperbolic knot for which the canonical component of $X(K)$ is the only component that contains the character of an irreducible representation. Then Conjecture 1.1 holds for K.*

This obviously follows from the stronger theorem stated below.

Theorem 4.4. *Suppose $K \subset S^3$ is as in Theorem 4.3. Then $\pi_1(S^3 \setminus K)$ does not surject onto the fundamental group of any other non-trivial knot complement.*

Proof. Assume to the contrary that $\phi : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K')$ is a surjection. It will be convenient to make use of the $\mathrm{PSL}(2, \mathbf{C})$ character variety. Let $Y_0(K)$ denote the canonical component of $Y(K)$, which as remarked upon in Section 2.3, has dimension 1.

K' cannot be a torus knot since $Y_0(K)$ contains the character of a faithful representation of $\pi_1(S^3 \setminus K)$ and $Y(C_{p,q})$ clearly contains no such character. That is to say $Y_0(K) \neq \phi^*(Y(C_{p,q}))$.

Theorem 3.1 handles the case when K' is hyperbolic. More precisely, taking $G = \pi_1(S^3 \setminus K)$, in the notation of Theorem 3.1, $k \leq 1$. Since, G surjects onto itself, we deduce that there can be no other knot group quotient.

Now assume that K' is a satellite knot. In this case, we use Theorem 2.2 to deduce that $\phi^*(Y(K'))$ coincides with $Y(K)$. However, if χ_ρ denotes the character of the faithful discrete representation on the canonical component $Y_0(K)$, then there is a character $\chi_\nu \in Y(K')$ with $\rho = \nu\phi$. But this is clearly impossible. □

By [19], when $n \geq 7$ is not divisible by 3 or $n \leq -1$ is not divisible by 3, the $(-2, 3, n)$-pretzel knot satisfies the hypothesis of Theorem 4.3. Hence we get.

Corollary 4.5. *Suppose that $n \geq 7$ is not divisible by 3, or $n \leq -1$ is not divisible by 3, then Conjecture 1.1 holds for the $(-2, 3, n)$-pretzel knot.*

5. Possible extension of Simon's Conjecture

We first state a possible extension of Simon's Conjecture for links. To that end, recall that a *boundary link* is a link whose components bound disjoint Seifert surfaces. Such a link (say with n components) has a fundamental group that surjects onto a non-abelian free group of rank n. A *homology boundary link* of n components is a link of n components whose fundamental group surjects onto a non-abelian free group of rank n.

Conjecture 5.1. *Let $L \subset S^3$ be a non-trivial link of $n \geq 2$ components. If $\pi_1(S^3 \setminus L)$ surjects onto infinitely many distinct link groups of n components, then L is a homology boundary link.*

This conjecture is motivated by Simon's conjecture for knots and the following observations:

If $n \geq 2$, then the trivial link of n-components has a fundamental group which is free of rank $n \geq 2$. Hence, it surjects onto all link groups that are generated by n elements. This argument can now be made by replacing the trivial link by a homology boundary link. In particular, since there are non-trivial boundary links of two components, the fundamental groups of such link complements will surject onto all two component two-bridge link groups.

Hyperbolic examples are easily constructed from this using [15] for example. Hence the group of any link with n components is the homomorphic image of the fundamental group of a hyperbolic link with n components.

As in Corollary 3.3, Theorem 3.1 provides information about the dimension of the character variety of a homology boundary link with $n \geq 2$ components (see also [7]):

Corollary 5.2. $\dim(X(L)) > n$ *for each homology boundary link L of $n \geq 2$ components.*

Proof. Let L be a homology boundary link of n components. The group of L surjects onto all n component link groups that are generated by n elements. As we note in the remark following this proof, infinitely many of these correspond to distinct hyperbolic link complements, and so $\dim(X(L)) \geq n$ by Lemma 2.1. Hence $\dim(X(L)) > n$ by Theorem 3.1. □

Remark. It is easy to see that there are infinitely many n-component hyperbolic links whose groups are generated by n-elements. Briefly, by Thurston's hyperbolization theorem for surface bundles [21, 30] a pseudo-Anosov pure braid with $n-1$ strings together with its axis forms a hyperbolic n-bridge link with n components. Moreover the group of such a link is generated by n elements. Since there are infinitely many conjugacy classes of pseudo-Anosov pure braids with $n-1$ strings, infinitely many distinct hyperbolic link complements can be obtained in this way. □

Another natural extension of Simon's Conjecture is.

Conjecture 5.3. *Let X be a knot exterior in a closed orientable three-manifold for which $H_1(X : \mathbf{Q}) \cong \mathbf{Q}$. Then $\pi_1(X)$ surjects onto only finitely many groups $\pi_1(X_i)$ where X_i is a knot exterior with $H_1(X_i : \mathbf{Q}) \cong \mathbf{Q}$.*

The condition on the rational homology is clearly a necessary condition (otherwise one can use surjections that factor through a non-abelian free group once again). Even here little seems known. Indeed, even for small manifolds as in Conjecture 5.3 we cannot make as much progress as in the case of S^3, since Theorem 2.2 of Kronheimer and Mrowka is not known to hold in this generality.

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